Journal of Engineering Physics and Thermophysics, Vol. 68, No. 3, 1995

RHEOLOGICAL FACTOR AND FAHRAEUS-LINDQVIST EFFECT

Z. P. Shul'man, L. V. Markova, and A. A. Makhanek

UDC 536.2.072:612.014.424.5

Human blood flow in a microvessel with regard for the Fahraeus-Lindquist effect is considered in an approximation of a two-layer model. The blood flow curve is described by the generalized equation of a nonlinear viscoplastic medium. Analytical expressions are derived for the volume blood flow velocity, effective blood viscosity, maximum flow velocity, and mean shear rate.

Space-time evolution of temperature fields in heated living biological tissue is basically specified by continuous heat removal due to blood flow. Biological-tissue heating depends in many respects on blood-flow intensity. In the thermophysical hyperthermia models correct allowance for the convective component is very difficult, since it is affected by many factors, namely, rheological, hemodynamic, and physiological. It is known [1] that blood behaves as a rheologically complicated medium as regards its nonlinear viscoplastic characteristics. In blood microvessels with a diameter less than 20 μm its flow is complicated by the Fahraeus-Lindquist (F-L) effect, i.e., the formation of a wall layer of plasma almost without erythrocytes (F-L layer) where the viscosity is lower [1, 2]. For channels with a diameter less than 50 μm, the F-L effect is rather substantial. According to data of various authors the width of the wall layer of plasma can reach 4–40 μm. Microrheological experiments have established the constant exchange of erythrocytes between the wall layer and the major flow: in the general case, the boundary of the wall layer is movable and, in some sense, conventional [2].

To determine analytically the width of the wall layer, Gazley has used a two-phase two-layer model [2]. Hematocrit in the wall layer was assumed equal to zero. From experimental data on a blood flow in small tubes the layer thickness was found equal to 4.45 μm independently of the tube diameter and hematocrit values. As direct measurements showed, Gazley had determined the lower width values of the wall layer. In actual fact, the width of the wall layer can be up to 38 μm [1] due to the aggregation and deformability of erythrocytes.

We shall consider an influence of the F-L effect on blood flow rate with allowance for the rheological factor. The volume blood flow rate per second will be determined using a two-layer model (Fig. 1). We use a rheological equation in the form of a generalized model of a nonlinear viscoplastic medium [3]:

$$\tau^{1/n} = \tau_0^{1/n} + (\mu_{\text{pl}}\dot{\gamma})^{1/m} . \tag{1}$$

It is generally accepted that plasma is a Newtonian fluid (at least for shear rates $\gamma > 5\ \text{sec}^{-1}$ [4]). Consequently, in the wall layer $\tau = \mu_0 \gamma$.

The volume plasma flow rate of the wall layer is determined from the expression

$$Q_{\text{p}} = \int\limits_{R-\delta}^{R} 2\pi r u_{\text{p}}(r)\, dr = \pi r^2 u_{\text{p}}(r) \Big|_{R-\delta}^{R} - \pi \int\limits_{u_\delta}^{0} r^2 du_{\text{p}}$$

with account of the wall condition $\bar{u}_{\text{p}}|_{r=R} = 0$ and $du_{\text{p}} = -\gamma dr$

$$Q_{\text{p}} = -Q^* + \pi \frac{\Delta P R^4}{8\mu_0 L}(1 - \varepsilon^4) ,$$

Academic-Scientific Complex "A. V. Luikov Heat and Mass Transfer Institute, Academy of Sciences of Belarus," Minsk, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 68, No. 3, pp. 416-426, May-June, 1995. Original article submitted December 2, 1993.

1062-0125/95/6803-0353$12.50

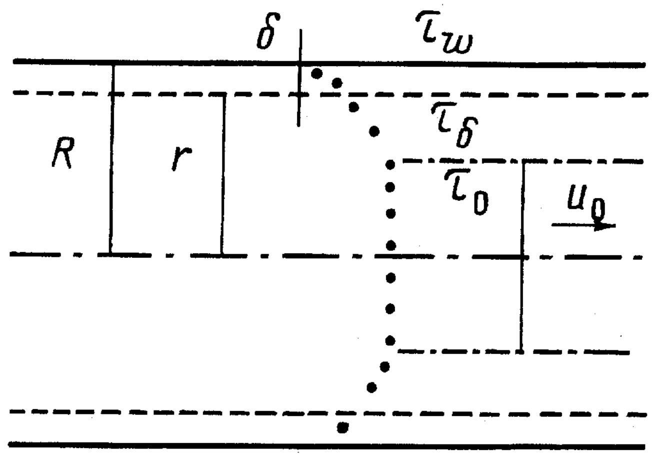


Fig. 1. Schematic of two-layer model of blood flow in vessel.

where

$$Q^* = \pi (R-\delta)^2 u_\delta ; \qquad u_\delta = u_p \Big|_{r=R-\delta} ; \quad \varepsilon = 1 - \frac{\delta}{R} .$$

From the equality

$$\int_{u_\delta}^{0} du_p = - \int_{R-\delta}^{R} \frac{\tau}{\mu_0} dr$$

and the balance condition of forces acting on a cylindrical element $2\pi r L \tau = \pi r^2 \Delta P$ or $\tau = \bar{r} \Delta P / 2L$ we obtain an expression for the boundary velocity of the F-L layer:

$$u_\delta = \frac{\Delta P R^2}{4 \mu_0 L} (1 - \varepsilon^2) \tag{2}$$

We designate $Q_{0p} = \pi \Delta P R^2 / 8 \mu_0 L$, then $Q^* = 2 Q_{0p} \varepsilon^2 (1 - \varepsilon^2)$ and

$$Q_p = Q_{0p} (1 - \varepsilon^2)^2 . \tag{3}$$

In the same manner we determine a blood flow rate but with use of (1):

$$Q_b = 2\pi \int_0^{R_0} r u_0 dr + 2\pi \int_{R_0}^{R-\delta} r u_b (r) \, dr ,$$

where $\bar{u}_0 = \bar{u}_b |_{\bar{r}=R_0}$ is the velocity of the quasisolid core; $R_0 = R \tau_2 / \tau_w \le R - \delta$; R_0 = is its radius; $\tau_w = \Delta P R / 2L$ is the shear rate on the blood vessel wall. Taking into account that $\bar{r} = (R - \delta) \tau / \tau_\delta$ where $\tau_\delta = \tau |_{\bar{r}=R-\delta}$ we arrive at

$$Q_b = Q^* + \frac{\pi (R-\delta)^3}{\tau_\delta} \int_{\tau_0}^{\tau_\delta} \tau^2 f(\tau) \, d\tau . \tag{4}$$

Here

$$f(\tau) = \frac{1}{\mu_{\mathrm{pl}}} (\tau^{1/n} - \tau_0^{1/n})^m = \frac{1}{\mu_{\mathrm{pl}}} \sum_{k=0}^{M} (-1)^k A_k \tau^{(m-k)/n} \tau_0^{k/n} ; \tag{5}$$

$$M = \begin{cases} m, \text{ for positive integers } m \text{ and } n; \\ \infty, \text{ for real positive } m \text{ and } n; \end{cases}$$

$$A_0 = 1; \quad A_k = \begin{cases} C_m^k, \text{ for integers } m > 0 \text{ and } n > 0 \\ m(m-1)\ldots(m-k+1)/k\,!, \text{ for real } m > 0 \text{ and } n > 0 \end{cases}.$$

Substitution of (5) into (4) and integration yield

$$M = \begin{cases} m, & \text{for positive integers } m \text{ and } n, \\ \infty, & \text{for real positive } m \text{ and } n; \end{cases}$$

$$A_0 = 1; \quad A_k = \begin{cases} C_m^k, & \text{for integers } m > 0 \text{ and } n > 0 \\ \dfrac{m(m-1)\ldots(m-k+1)}{k\,!}, & \text{for real } m > 0 \text{ and } n > 0. \end{cases}$$

$$Q_b = Q^* + Q_{0b}\frac{4}{3 + m/n}\varepsilon^{3+m/n}\tau^{m/n-1}\varphi_{m,n}(\zeta_\delta), \tag{6}$$

where

$$Q_{0b} = \frac{\pi \Delta P R^4}{8\mu_{pl}}; \quad \zeta_\delta = \frac{\tau_0}{\tau_\delta} = \frac{1}{\varepsilon}\frac{\tau_0}{\tau_w} = \frac{1}{\varepsilon}\zeta_w;$$

$$\varphi_{m,n}(\zeta) = \sum_{k=0}^{M}(-1)^k\frac{m+3n}{m+3n-k}A_k\zeta^{k/n}\left[1 - \zeta^{(m+3n-k)/n}\right]. \tag{7}$$

At $m = n$

$$Q_b = Q^* + Q_{0b}\,\varepsilon^4\varphi_{m=n}(\zeta_\delta). \tag{8}$$

The function $\varphi_{m,n}$ possesses the following properties:

$$0 \le \varphi_{m,n}(\zeta) \le 1 \quad \text{for} \quad 0 \le \zeta \le 1;$$

$$\varphi_{m,n}(0) = 1; \quad \varphi_{m,n}(1) = 0.$$

In the special cases $m = n = 1$; $m = n = 2$, and $m = n = 3$ it has the form

$$\varphi_{m=n=1}(\zeta) = 1 - \frac{4}{3}\zeta + \frac{1}{3}\zeta^4, \tag{9}$$

$$\varphi_{m=n=2}(\zeta) = 1 - \frac{16}{7}\zeta^{1/2} + \frac{4}{3}\zeta - \frac{1}{21}\zeta^4, \tag{10}$$

$$\varphi_{m=n=3}(\zeta) = 1 - \frac{36}{11}\zeta^{1/3} + \frac{36}{10}\zeta^{2/3} - \frac{4}{3}\zeta + \frac{1}{165}\zeta^4. \tag{11}$$

Thus, the total flow rate of the plasma of the wall layer and the erythrocyte suspension in the plasma $Q = Q_p + Q_b$ with an account of (3) and (6) is

$$Q = Q_{0p}(1 - \varepsilon^4) + Q_{0b}\,\varepsilon^{3+(m/n)}\tau_w^{(m/n)-1}\varphi_{m,n}(\zeta_\delta). \tag{12}$$

In the special case $m = n$, we obtain

$$Q = Q_{0p}(1 - \varepsilon^4) + Q_{0b}\,\varepsilon^4\,\varphi_{m=n}(\zeta_\delta)\,. \tag{13}$$

In further analysis of the F-L effect, without loss of generality, we shall proceed from expression (13), i.e., $m = n$ in model (1). We shall consider at first some limiting cases.

Physical correctness of the result obtained for the plasma flow rate of the F-L layer is most manifested in the limiting situations when $\varepsilon = 0$, i.e., $\delta = R$ (pure plasma flow in a blood vessel) and $\varepsilon = 1$ (the F-L layer is absent). In the former case, the plasma flow rate pertains to the Poiseulle type; in the latter case, $Q_p = 0$. The matter is somewhat more difficult in the case of the expressions (6) and (8) for the blood flow rate. If at $\varepsilon = 1$ one obtains an ordinary expression for the viscoplastic fluid flow rate (equivalent to that given in [3]) then in the case of $\varepsilon = 0$ formulas (6) and (8) do not "work," since the blood flow rate is negative because (6) and (8) are derived for the condition $R - \delta \geq R_0$, i.e., $\tau_\delta = \tau_w\delta \geq \tau_0$. Consequently, $\delta \geq \tau_0/\tau_w$ and

$$\delta \leq \delta_{\max} = R\left(1 - \frac{\tau_0}{\tau_w}\right). \tag{14}$$

Thus, the maximum possible thickness of the F-L layer $\delta_{\max}$ must increase with an increase of τ_w and with a decrease of τ_0. It is self-evident that the actual thickness of the wall layer of plasma is not obligatorily equal to $\delta_{\max}$. But its possible values are upper-bounded in accordance with expression (14).

In [1], for a blood vessel with a diameter of 190 μm the thickness of the plasma layer was observed to increase near the vessel wall with increasing τ_w until τ_w attained 0.2–0.4 N/m^2. On further increasing τ_w, the thickness of the F-L layer began to decrease up to 0 at $\tau_w = 15$ N/m^2. Its maximum value attained $\delta_{\max} = 38$ μm. If in this case a limiting value corresponding to expression (14) was attained, then τ_0 was evaluated as:

$$\tau_0 = \tau_w(1 - \delta_{\max}/R) \quad \text{and} \quad \tau_0 = 0.3\,(1 - 38/95) \simeq 0.18 \ \text{N/m}^2.$$

In [1] it is noted that τ_0 does not exceed 0.05 N/m^2 for human blood. Thus, the τ_0 estimate obtained is excessive. Substituting the upper limit $\tau_0 = 0.05$ into (14) yields $\delta_{\max} = 95(1 - 0.05/0.3) \simeq 80$ μm which is much higher than the value obtained in experiments. Consequently, the limiting thicknesses of the F-L layer are substantially lower than those obtained from expression (14), i.e., for human blood $R_0 < R - \delta$.

One of the factors governing this inequality is variability of the erythrocyte concentration over the vessel section [1]. A decrease of the concentration (and, consequently, of τ_0), from some maximum value on the axis to zero in the region of the F-L layer boundary, must entail a decrease in the effective (mean over the vessel section) limiting shear stress as compared with its local values in the quasisolid core and in the flow region adjacent to it.

In the limiting case $\delta = \delta_{\max}$, with $\delta_{\max}$ calculated by (14), the total flow rate of plasma and blood is

$$Q = Q_{0p}(1 - \varepsilon_{\min}^4)\,, \quad \text{and} \quad \mu_e = \mu_0/(1 - \varepsilon_{\min}^4)\,. \tag{15}$$

Here $\varepsilon_{\min} = (1 - \delta_{\max})/R$; $\mu_e = \pi R^4 \Delta P/8QL$ is the total effective viscosity of human plasma and blood.

Thus, at $\delta = \delta_{\max}$ the total flow rate of blood and plasma in the wall layer as well as their effective viscosity are completely determined by the plasma viscosity and the parameter $\varepsilon_{\min} = \xi_w$, i.e., by the maximum relative width of the quasisolid core.

According to (12), (13), at $\mu_0 < \mu_{\mathrm{pl}}$ (for blood this inequality is always fulfilled) the F-L effect increases the blood flow rate against that in the absence of a wall layer of plasma.

In fact, in the latter case $Q_0 = Q_{0b}\varphi_{m,n}(\xi_w)$ and

$$\frac{Q}{Q_0} = \frac{\mu_{\mathrm{pl}}}{\mu_0}\,\frac{1 - \varepsilon^4}{\varphi_{m,n}(\zeta_w)} + \varepsilon^{3 + m/n}\,\frac{\varphi_{m,n}(\zeta_w/\varepsilon)}{\varphi_{m,n}(\zeta_w)}\,. \tag{16}$$

The derivative of this expression with respect to ε is

$$\left(\frac{Q}{Q_0}\right)'_\varepsilon = \frac{\varepsilon^2}{\varphi_{m,n}(\zeta_w)}\left[-4\frac{\mu_{\mathrm{pl}}}{\mu_0}\varepsilon + \frac{m+3n}{n}(\varepsilon^{1/n} - \zeta_w^{1/n})^m\right] \tag{17}$$

and at $m = n$

$$\left(\frac{Q}{Q_0}\right)'_\varepsilon = \frac{4\varepsilon^2}{\varphi_{m=n}(\zeta_w)}\left[-\frac{\mu_{\mathrm{pl}}}{\mu_0}\varepsilon + (\varepsilon^{1/n} - \zeta_w^{1/n})^n\right] < 0$$

for any $\varepsilon > \varepsilon_{\min} = \xi_w$ and $\mu_{\mathrm{pl}} > \mu_0$.

Thus, at $n \geq m$ and $\mu_{\mathrm{pl}} > \mu_0$ the ratio Q/Q_0 always decreases with an increase of ε or, what is the same, with a decrease of δ.

At $m > 0$ combinations of m, n and μ_{pl}/μ_0 are formally possible such that, as follows from (17), Q/Q_0 does not increase but, on the contrary, decreases with increase of a width of the F-L layer even at $\mu_0 < \mu_{\mathrm{pl}}$. This is presumably associated with the fact that the obtained expression (12) is approximate. It is also possible that in reality the conditions (the combinations of m, n, and μ_{pl}/μ_0) at which Q/Q_0 could decrease with a δ rise are physically unreliazable. One more reason, in our opinion, is the neglect, in calculating derivative (17), of the wall shear stress dependence of the wall layer width of plasma. Thus, expression (17) should be considered approximate even within the framework of the two-layer model of a microvessel blood flow.

In [1], an expression is given for the asymptotic ($\delta << R$) viscosity of an erythrocyte suspension in the presence of the F-L effect. Using our notation it is written as

$$\mu_{\mathrm{e}} = \mu_0\left[\frac{\mu_0}{\mu_{\mathrm{pl}}}\left(1 + 4\frac{\delta}{R}\right) - 4\frac{\delta}{R}\right]^{-1}. \tag{18}$$

It is easy to obtain this relation from (13) at $\varphi_{m,n}(\xi_\delta) = 1$. In fact, dividing the right- and left-hand sides of (13) by $\pi R^4\Delta P/8L$, we arrive at

$$\frac{1}{\mu_{\mathrm{e}}} = \frac{1-\varepsilon^4}{\mu_0} + \frac{\varepsilon^4}{\mu_{\mathrm{pl}}}\varphi_{m,n}(\zeta_\delta). \tag{19}$$

Since in accordance with the condition $\delta << R$, then $\varepsilon^4 \simeq 1 - 4\delta/R$ and $(1 - 4\delta/R)/\mu_{\mathrm{pl}} = -(4\delta/R)/\mu_0 + 1/\mu_{\mathrm{e}}$ or $1/\mu_{\mathrm{pl}} \simeq (1 + 4\delta/R)/\mu_{\mathrm{e}} - (4\delta/R)/\mu_0$, whence (18) follows.

In works [2, 5] an expression is derived which can be reduced exactly to the form of (19) but at $\varphi_{m,n} = 1$ (blood was considered a Newtonian fluid). Thus, (12) generalizes the known theoretical results of modeling the total flow rate of plasma of the F-L layer and blood.

Figure 2 shows the effective blood viscosity versus the vessel diameter. Curve 1 represents its asymptotic (at large γ) values [1], while curve 2 shows the viscosities at "physiological" shear rates [7].

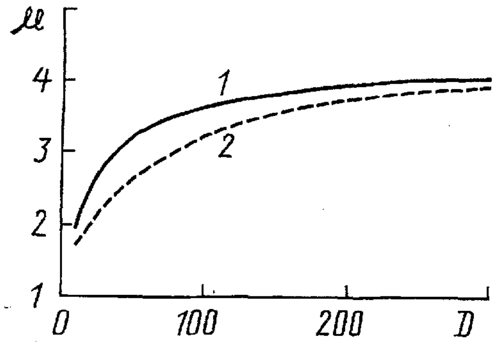
We now evaluate the ratio of the erythrocyte velocity to the plasma velocity of the wall layer. At first we shall analyze the maximum u_0 and u_δ values. The value of u_δ is given by formula (2). For u_0 we have

$$\int_{u_0}^{u_\delta} du_{\mathrm{b}} = -\int_{R_0}^{R-\delta} f(\tau)\,dr\,,$$

Whence with regard for (5) we obtain

$$u_0 = u_\delta + \frac{n}{m+n}\frac{R-\delta}{\mu_{\mathrm{pl}}}\tau_\delta^{m/n}\psi_{m,n}(\zeta_\delta)\,, \tag{20}$$

where



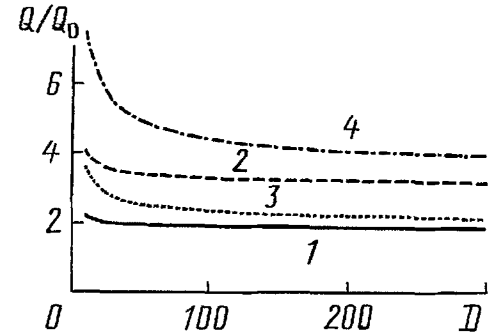


Fig. 2. Viscosity versus tube diameter: 1) data [1]; 2) data [7]. μ, cP; D, μm.

Fig. 3. Relative influence of F-L effect on blood flow rate in tube: 1) $\delta/R = 0.1$, $m = n = 2$; 2) $\delta/R = 0.4$, $m = n = 2$; 3) $\delta/R = 0.1$, $m = n = 3$; 4) $\delta/R = 0.4$, $m = n = 3$.

$$\psi_{m,n}(\zeta) = \sum_{k=0}^{M} (-1)^k \frac{m+n}{m+n-k} A_k \zeta^{k/n} [1 - \zeta^{(m+n-k)/k}]. \tag{21}$$

The function $\psi_{m,n}$, like $\varphi_{m,n}$, posseses the following properties

$$0 \le \psi_{m,n}(\zeta) \le 1 \quad \text{for} \quad 0 \le \zeta \le 1;$$

$$\psi_{m,n}(0) = 1; \quad \psi_{m,n}(1) = 0.$$

In the special cases $m = n = 1$, $m = n = 2$, and $m = n = 3$ it has the form

$$\psi_{m=n=1}(\zeta) = 1 - 2\zeta + \zeta^2, \tag{22}$$

$$\psi_{m=n=2}(\zeta) = 1 - \frac{4}{3}\zeta^{1/2} + 2\zeta - \frac{1}{3}\zeta^2, \tag{23}$$

$$\psi_{m=n=3}(\zeta) = 1 - \frac{18}{5}\zeta^{1/3} + \frac{9}{2}\zeta^{2/3} - 2\zeta + \frac{1}{10}\zeta^2. \tag{24}$$

Thus,

$$\frac{u_0}{u_\delta} = 1 + \frac{n}{m+n} \frac{\mu_0}{\mu_{\rm pl}} \frac{\varepsilon^{1+(m/n)}}{1-\varepsilon^2} \tau_w^{(m/n)-1} \psi_{m,n}(\zeta_\delta). \tag{25}$$

At $m = n$

$$\frac{u_0}{u_\delta} = 1 + \frac{1}{2} \frac{\mu_0}{\mu_{\rm pl}} \frac{\varepsilon^2}{1-\varepsilon^2} \psi_{m=n}(\zeta_\delta). \tag{26}$$

Analogously, for the ratio of mean velocities of blood and plasma of the wall layer we write

$$\frac{\bar{u}_{\rm b}}{\bar{u}_{\rm p}} = \frac{Q_{\rm b}/\pi (R-\delta)^2}{Q_{\rm p}/\pi [R^2 - (R-\delta)^2]}.$$

Whence with regard for (3) and (6) we derive

$$\frac{\bar{u}_b}{\bar{u}_p} = 2\left[1 + \frac{2\tau_w^{m/n-1}}{3 + m/n}\frac{\mu_0}{\mu_{pl}}\frac{\varepsilon^{1+m/n}}{1-\varepsilon^2}\psi_{m,n}(\zeta_\delta)\right]. \tag{27}$$

If $m = n$, then

$$\frac{\bar{u}_b}{\bar{u}_p} = 2\left[1 + \frac{1}{2}\frac{\mu_0}{\mu_{pl}}\frac{\varepsilon^2}{1-\varepsilon^2}\psi_{m=n}(\zeta_\delta)\right]. \tag{28}$$

We write for $m = n$

$$(\bar{u}_b/\bar{u}_p)_{\min} = 2\left[1 + \frac{1}{2}\frac{\mu_0}{\mu_{pl}}\frac{\zeta_w^2}{1-\zeta_w^2}\psi_{m=n}(\zeta_\delta)\right] \geq 2\,;$$

$$(\bar{u}_b/\bar{u}_p)_{\max} = \infty\,.$$

Expressions (25)-(28) qualitatively correctly illustrate the known fact that the erythrocyte velocity exceeds the plasma filtration velocity of the F-L layer [1, 6]. Moreover if, according to [1], a ratio of their mean velocities in tubes with a diameter of 6–10 μm is $\bar{u}_b/\bar{u}_p \simeq 1.1-1.3$, then formula (28) gives a value which is always in excess of 2. For instance, at $\mu_0/\mu_{pl} \simeq 1/2$, $\varphi \simeq 1$ and $\delta/R \simeq 3 = 0.6$ we have $\bar{u}_b/\bar{u}_{pl} \simeq 2.05$, which is markedly higher than the observed values.

Expression (26) satisfies the given ratio of velocities. Thus, with the assumptions made, $u_0/u_\delta \simeq 1.03$. However, it characterizes, as already mentioned, the ratio of the maximum (not mean with respect to the vessel section) velocities of erythrocytes and plasma of the wall layer. The situation is also similar when $n \neq m$ in (1). Therefore it may be concluded that expressions (12) and (13), which were obtained for the total flow rate of an erythrocyte suspension in plasma and the plasma of the wall layer with an account of the F-L effect, should be considered only as a first approximation. In particular, these relations "work" foorly when the vessel diameter is comparable with erythrocyte size.

We now evaluate the drag coefficient $C_f = \tau_w/(\rho\bar{u}^2/2)$ at $n = m$ with regard for the F-L effect for each component of the flow and their sum.

Expressing τ_w in terms of Q_p, Q_b, and Q in formulas (3), (7), and (8) and $\bar{u}$ in terms of the Reynolds number $\mathrm{Re} = \rho l\bar{u}/\mu$, where l is the characteristic size, ρ is the density, we arrive at

$$C_{fp} = \frac{16}{\mathrm{Re}_p}\frac{1-\varepsilon}{1+\varepsilon};\quad C_{fb} = \frac{16}{\mathrm{Re}_b}\frac{\varepsilon^4}{\frac{\mu_{pl}}{\mu_0}(1-\varepsilon^4) + \varepsilon^4\varphi_{m,n}(\zeta_\delta)};\quad C_f = \frac{16}{\mathrm{Re}}. \tag{29}$$

Here

$$\mathrm{Re}_p = \frac{\rho_p\bar{u}_p 2\delta}{\mu_0} = K\frac{\rho_p}{\bar{\rho}}\mu_0^{-2}(1-\varepsilon)(1-\varepsilon^2)\,; \tag{30}$$

$$\mathrm{Re}_b = \frac{\rho_b\bar{u}_{b2}(R-\delta)}{\mu_a} = K\frac{\rho_b}{\bar{\rho}}\mu_{pl}^{-2}\varepsilon\left[2\frac{\mu_{pl}}{\mu_0}(1-\varepsilon^2) + \varepsilon^2\varphi_{m,n}(\zeta_\delta)\right]^2; \tag{31}$$

$$\mathrm{Re} = \frac{\bar{\rho}\,\bar{u}\,2R}{\mu_a} = K\mu_{pl}^{-2}\left[\frac{\mu_{pl}}{\mu_0}(1-\varepsilon^4) + \varepsilon^4\varphi_{m=n}(\zeta_\delta)\right]^2; \tag{32}$$

$$K = \frac{1}{4}\bar{\rho} R^3 \frac{\Delta P}{L}; \quad \bar{\rho} = \rho_p (1 - \varepsilon^2) + \rho_p \varepsilon^2 . \tag{33}$$

One of the causes of overestimation of the mean erythrocyte velocity as compared with the mean plasma velocity in the wall layer as calculated by formulas (27) and (28) is apparently the fact that the actual values of the coefficients for blood and plasma should be somewhat higher than those calculated by (29)

$$C^*_{fp} = C_{fp} + \Delta C_{fp}; \quad C^*_{fb} = C_{fb} + \Delta C_{fb}; \quad \Delta C_{fp} > 0; \quad \Delta C_{fb} > 0,$$

where $\Delta C_{fb} > \Delta C_{fp}$ because of additional energy losses due to destruction of erythrocyte aggregates at their inlet to narrower blood vessels (especially capillaries) from the main vessel as well as due to their interaction with the endothelium of capillary walls and deformation and rotation of erythrocytes which results in the emergence of a pseudoturbulent flow [7].

Since information found in literature about the mean, with respect to the vessel section, flow velocity $\bar{u}$ and shear rate $<\dot{\gamma}>$ sometimes pertains to different vessels, it is of interest to derive analytical expressions for these parameters with and without allowance for the F-L effect and to compare their estimates with observation data.

For the mean flow velocity, as follows from (13):

$$\bar{u} = \frac{1}{4}\tau_w R \left[\frac{1 - \varepsilon^4}{\mu_0} + \frac{\varepsilon^4}{\mu_{\text{pl}}} \varphi_{m=n}(\tau_0/\varepsilon\,\tau_w) \right], \tag{34}$$

and in the absence of the F-L effect

$$\bar{u}_0 = (\tau_w R/4\mu_{\text{pl}})\, \varphi_{m=n}(\tau_0/\tau_w) . \tag{35}$$

The mean shear rate is

$$\langle \dot{\gamma} \rangle = \frac{1}{R}\int_0^R \dot{\gamma} dr = \frac{1}{R}\int_R^{R-\delta} \dot{\gamma}_{\text{b}} dr + \frac{1}{R}\int_{R-\delta}^R \dot{\gamma}_{\text{p}} dr .$$

Integration yields

$$\langle \dot{\gamma} \rangle = \frac{\tau_w}{2} \left[\frac{1 - \varepsilon^2}{\mu_0} + \tau_w^{m/n} \frac{\varepsilon^{m/n+1}}{\mu_{\text{pl}}} \psi_{m,n}(\zeta_\delta) \right] . \tag{36}$$

In the special case of $m = n$

$$\langle \dot{\gamma} \rangle = \frac{\tau_w}{2} \left[\frac{1 - \varepsilon^2}{\mu_0} + \frac{\varepsilon^2}{\mu_{\text{pl}}} \psi_{m=n}(\zeta_\delta) \right], \tag{37}$$

and in the absence of the F-L effect

$$\langle \dot{\gamma} \rangle_0 = \frac{\tau_w}{2\mu_{\text{pl}}} \psi_{m=n}(\zeta_w) . \tag{38}$$

In [1], characteristic values are given for the main parameters of human blood flow in different vessels. Some of them are contained in Table 1. We shall evaluate the mean flow velocity and mean shear rate for the case considered above when in a blood vessel with a diameter of 190 μm at $\tau_m = 0.3$ N/m^2 an F-L layer with thickness $\delta = 38$ μm is observed. At $m = n = 2$ and $m = n = 3$ the mean velocities calculated for these data are $\bar{u}_{m=n=2} \simeq 0.51$ cm/sec and $\bar{u}_{m=n=3} \simeq 0.50$ cm/sec, i.e., using any of these models we obtain a mean velocity equal, approximately, to 0.5 cm/sec, which agrees with the data from [2] (see Table 1). Analogously, for the mean shear

TABLE 1. Some Parameters of Human Blood Flow

Vessel	D, cm	L, cm	$\bar{u}$, cm/sec	$\dot{\gamma}$, 1/sec	Re
Aorta	1.6–3.3	80	60–30	100	1200–5800
Large arteries	0.6–0.1	40–20	20	400	1000–100
Small arteries, arterioles	0.1–0.02	5–0.2	10–0.2	>100	10–0.01
Capillaries	0.0005–0.001	0.1	0.05–0.07	400	0.001–0.003
Venules, small veins	0.02–0.2	0.2–1	0.1–1	~100	0.01–1
Large veins	0.5–1	10–30	10–20	100	100–600
Hollow veins	2	50	10–20	50	600–1000

TABLE 2. Mean Flow Velocity and Shear Rate in a Capillary at Different Thicknesses of the Wall Layer of Human Plasma

$\varepsilon_{\min}$	δ, μm	$\bar{u}$, cm/sec	$\dot{\gamma}$, 1/sec
0.90	0.50	0.23	14200
0.95	0.25	0.16	950
0.97	0.15	0.10	590
0.98	0.10	0.065	320

rates we obtain $<\dot{\gamma}>_{m=n=2} \simeq 96$; $<\dot{\gamma}>_{m=n=3} \simeq 89$, which is somewhat smaller than the lower possible limit (100 $\sec^{-1}$) of this parameter in arterioles and venules.

But it is more difficult to evaluate $\bar{u}$ and $<\dot{\gamma}>$ for a capillary. Thus, if we assume that the capillary length is $L = 1$, the radius is $R = 5\ \mu$m, and the pressure drop at its ends is $\Delta P = 20$ mm Hg (or 2666 N/m^2), then at $\delta/R \simeq 0.1$ ($\varepsilon = 0.9$) we obtain $\tau_w \simeq 6.67$ N/m^2, $\xi_\delta \simeq 0.001/(6.67 \cdot 0.9) \simeq 1.7 \cdot 10^{-4}$ and $\bar{u}_{m=n=2} \simeq 0.49$ cm/sec instead of 0.05–0.07, i.e., by an order higher. Here $<\dot{\gamma}> \simeq 1890$ which is also substantially higher than the value (400 $\sec^{-1}$) reported in [1].

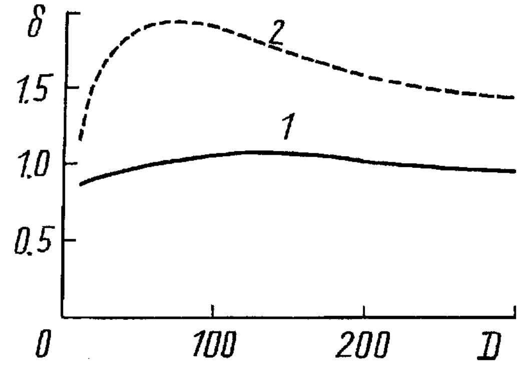
This estimate confirms once more that formulas (12), and (13) are inapplicable for blood flow in a capillary. If it is evaluated from (15) for flow rates of blood and plasma at $\varepsilon = \varepsilon_{\min}$, then for $\varepsilon_{\min}$ within the limits 0.9–0.98 (or $\delta = 0.5$–$0.1\ \mu$m) one may obtain the $\bar{u}$ and $<\dot{\gamma}>$ values given in Table 2. As is seen from the table, the results highly depend on the relative width of the wall layer of plasma, and if it is sufficiently small, one may obtain filtration velocities and shear rates in the capillary which are close to the observed values.

Numerical estimation of the influence of the F-L effect on the blood flow rate involves some difficulties since the physiologically justified parameters $\Delta P/L$, δ, R, μ_0 as well the parameters τ_0, μ_{pl}, m and n of the model (1) must be accounted for. At the same time some dependence exists between δ, R and τ_w (or $\Delta P/L$) which is not described functionally. Therefore we shall evaluate only the limiting values of Q/Q_0, thus greatly simplifying this problem. From (16) at $m = n$ we have

$$(Q/Q_0)_{\min} = 1 \quad \text{at} \quad \varepsilon = 1\,,$$

$$(Q/Q_0)_{\max} = \mu_{\text{pl}}\,(1 - \zeta_w^4)/(\mu_0\,\varphi_{m=n}(\zeta_w)) \quad \text{at} \quad \varepsilon = \zeta_w\,.$$

Figure 3 shows the results of $(Q/Q_0)_{\max}$ calculation at $\Delta P = 2666$ N/m^2 (20 mm Hg), $L = 0.01$ m, $\mu_0 = 1.3$ cP (curves 1, 2 $m = n = 2$ and 3, 4 $m = n = 3$). Since the parameter μ_{pl} of the rheological model (1) represents an asymptotic (at an infinitely high shear rate) value of the apparent Newtonian viscosity, then in our calculations μ_{pl}



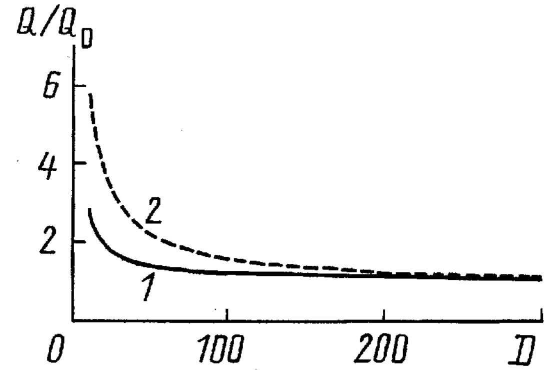


Fig. 4. F-L layer thickness calculated by formula (39): 1) data [1]; 2) data [7]. δ, μm.

Fig. 5. Relative influence of F-L effect on a blood flow rate in tube calculated by the formula $Q/Q_0 = \mu_{\mathrm{pl}}/\mu_{\mathrm{a}}$: 1) data [1]; 2) data [7].

= 4.3 cP (see Fig. 2). Here it is assumed that the relative thickness of the F-L layer is independent of the vessel diameter, i.e., δ/R = const. Consideration has been made of the variants with $\delta/R = 0.1$ and $\delta/R = 0.4$ (curves 1, 3 and 2, 4, respectively).

The relative influence of the F-L effect, as we might expect, is stronger (Fig. 3) in vessels with a small diameter, especially if $D < 30\,\mu$m. Here the choice of a rheological model (in our case $m = n = 2$ or $m = n = 3$) becomes more essential. The assumption made about δ/R constancy is, virtually, a very rough one and is not fulfilled in a wide range of vessel diameters. As already mentioned, starting from some τ_w value, the thickness of the F-L layer decreases with an increase of τ_w (and, consequently, with D rise if $\Delta P/L$ = const) until it vanishes. Nevertheless, even within the framework of this approximation the flow rate characteristics of blood are observed to come closer in the presence and absence of the F-L effect with increase in the vessel diameter. Here at the limit $\lim\limits_{\tau_w \to \infty} (Q/Q_0) = (\mu_{\mathrm{pl}}/\mu_0)(1 - \varepsilon^4) + \varepsilon^4$.

Using the asymptotic viscosity values shown in Fig. 2 and assuming that $\varphi_{m=n}(\xi_w) \simeq 1$, which corresponds to infinitely large shear rates, we may evaluate the change in the wall layer width of the plasma with increase in the vessel diameter. Thus, from formula (19) we obtain

$$\delta = R\left(1 - \sqrt[4]{\left(\frac{1 - \mu_0/\mu_{\mathrm{a}}}{1 - \mu_0/\mu_{\mathrm{pl}}}\right)}\right). \tag{39}$$

Substituting μ_{a} from [1, 7] into (39) (see Fig. 2) yields δ values shown in Fig. 4, where the thickness of the F-L layer is seen at first to increase with D and then at sufficiently large D to decrease. Here, maximum δ values lie in the range $D = 60-80\,\mu$m, though the maximum on curve 1 is substantially less pronounced than on curve 2. In the both cases the calculated thickness of the F-L layer is less than $2\,\mu$m, which can be attributed to a sufficiently large value of τ_w.

Figure 5 shows the influence of the F-L effect on the blood flow rate. Q/Q_0 has been calculated by the formula $Q/Q_0 = \mu_{\mathrm{pl}}/\mu_{\mathrm{a}}$ obtained from (16) in the approximation $\varphi_{m=n}(\xi_w) \simeq 1$. The μ_{a} values at different D are taken from Fig. 2. As follows from Fig. 5, allowance for the actually measured viscosity μ_{a} yields a substantially smaller Q/Q_0 than in Fig. 3. However, as in the previous case, the F-L effect is considerably stronger in vessels with a diameter smaller than 70 μm. The difference between curves 1 and 2 in Figs. 2, 4, and 5 could be due to differences in some of the rheological parameters of the investigated blood, e.g., τ_0, as well as to a different levels of $\Delta P/L$ in these experiments (moreover, it is unknown whether the pressure gradient was the same in vessels of different diameters or not). Curves 1 and 2 in Fig. 4 are obtained in an approximation of "high" shear rates but in actual fact curve 2 corresponds, as noted in [7], to "physiological" flow velocities and this means that $\dot{\gamma} \simeq 100-400$ sec^{-1}. In this case, δ values should be larger than in curve 1, which is qualitatively confirmed by the δ value calculated by formula (40). The actual width of the wall layer of plasma in the case of curve 2 should be even larger than the calculated one since at small $\dot{\gamma}$ the neglected function $\varphi_{m=n}(\xi_w)$ may be substantially smaller than curve 1.

In conclusion, expression (13) obtained with regard for the F-L effect qualitatively correctly describes the blood flow rate through a microvessel. A stronger influence of the F-L effect in the vessels with diameters less than 100 μm as well as a dependence of a width of the wall plasma layer on a vessel diameter are confirmed quantitatively. It is shown that the choice of nonlinearity parameters of the rheological flow model acquires a particular significance in describing blood flow in arterioles and venules with a diameter less than 100 μm.

The authors thank the Foundation of Fundamental Investigations of Belarus for financing the work (contract B3-222).

NOTATION

τ, shear stress; τ_0, limiting shear stress (yield stress); $\mu_{\rm pl}$, analog of plastic viscosity; μ_0, human plasma viscosity; $\dot{\gamma}$, shear rate; m, n, nonlinearity parameters of the rheological model; R, vessel radius; δ, thickness of the wall layer of human plasma; r, current radius; $u_{\rm p}$, velocity of human plasma; $\Delta P/L$, pressure gradient in vessel; $Q_{\rm p}$, volume flow rate of human plasma; $Q_{\rm b}$, volume flow rate of blood; u_0, quasisolid core velocity; τ_δ, shear stress at the plasma-blood interface; $\tau_w = \tau_0/\tau_w$, relative thickness of the quasisolid core of a viscoplastic flow; $\mu_{\rm e}$, total effective viscosity of human plasma and blood; f, φ, ψ, functions accounting for blood plasticity; $\bar{u}_{\rm b}$, mean velocity of blood flow; $\bar{u}_{\rm p}$, mean velocity of plasma flow; c_f, drag coefficient; Re, Reynolds number; l, characteristic size; ρ, density; $\bar{u}$, mean velocity of blood flow and the wall plasma layer; $\langle\dot{\gamma}\rangle$, mean shear rate; $\mu_{\rm a}$, asymptotic value of the viscosity at a high shear rate.

REFERENCES

1. V. A. Levtov, S. A. Regirer, and N. Kh. Shadrina, Blood Rheology [in Russian], Moscow (1982).
2. C. Gazley, in: Heat and Mass Transfer [in Russian], Collected Papers of the All-Union Conference, Minsk, Vol. 9. Pt. 1 (1972), pp. 3-22.
3. B. M. Smol'skii, Z. P. Zhul'man, and V. M. Gorislavets, Rheodynamics and Heat Transfer in Nonlinear Viscoplastic Materials [in Russian], Minsk (1970).
4. N. N. Firsov, Macro- and Micro-Rheology of Human Blood in Norm and Pathology [in Russian], Doctoral Dissertation in Biological Sciences, Moscow (1983).
5. R. H. Haynes, Amer. J. Physiology, **198**, No. 6, 1195 (1960).
6. G. M. Kostin, Change in Rheological Properties of Human Blood in the Acute Poisoning Cases [in Russian], Candidate's Dissertation in Medicine, Moscow (1976).
7. B. Folkov and E. Neil, Blood Circulation, New York, Oxford University Press, Londow-Toronto (1971).